Kashiwara Crystals of Type A in Low Rank

Ola Amara-Omari, Malka Schaps

Bar-Ilan University

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# Table of Contents

1. Affine Lie algebras of Type $A$
2. Crystals of Type $A$
3. Canonical basis
4. Symmetric Crystals
5. Bibliography
The Problem

- The irreducible modules for the symmetric groups over \( \mathbb{C} \) are labelled by partitions.
- Over a field \( F \) of characteristic \( p \), the irreducible modules of the tower of group algebras are labelled by \( p \)-regular partitions.
- Take a element \( \xi \) in a field \( k \) with \( 1 + \xi + \cdots + \xi^{e-1} = 0 \). For the tower of cyclotomic Hecke algebras over \( K \), the irreducible modules are labelled by \( e \)-regular multipartitions.

The problem: For \( e > 2 \), we have only a recursive algorithm for constructing \( e \)-regular multipartitions.
Affine Lie Algebras of Type A

- $\mathcal{G}$ - an affine Lie algebra of type A,
- Dynkin diagram a circle,
- $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ - fundamental weights, $\ell = e - 1$
- $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ - simple roots,
- $\delta = \sum \alpha_i$ - the null root.
- $Q_+ = \{\alpha | \alpha = \sum c_i \alpha_i\}$, with content $(c_0, c_1, \ldots, c_\ell)$,
- A corank 1 Cartan matrix

$$A = \begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & \cdots & 0 & -1 & 2
\end{bmatrix}$$
Kashiwara crystals

Let $e_i, f_i, h_i, i = 0, 1, \ldots, \ell, c$ be a Chevalley basis

Let $\Lambda = a_0 \Lambda_0 + \ldots a_\ell \Lambda_\ell, a_i \in \mathbb{Z}_+$

Let $V(\Lambda)$ be a highest weight representation generated by the $f_i$ from $u_\emptyset$ of weight $\Lambda$

Let $P(\Lambda)$ be the sets of weights of weight spaces of $V(\Lambda)$

Let $\max(\Lambda)$ be the set of weights $\eta \in P(\Lambda)$ such that $\eta + \delta \notin P(\Lambda)$.

A Kashiwara crystal $B(\Lambda)$ is

- a labeling of the basis of $V(\Lambda)$
- operations $\tilde{e}_i$ and $\tilde{f}_i$,
- functions $\phi_i, \epsilon_i$ measuring the distance to the end of the local $i$-string.
There are three important versions of the Kashiwara crystal in type A:

- by e-regular multipartitions
- by Littelmann paths,
- and by canonical basis elements, in a space called Fock space, with coefficients which are $\nu$-polynomials
There are three important versions of the Kashiwara crystal in type $A$:

- by $e$-regular multipartitions
- by Littelmann paths,
- and by canonical basis elements, in a space called Fock space, with coefficients which are $v$-polynomials

Our general research program concerns the combinatorial relations among all three, but for this talk, we focus on the possibility of passing directly between the $e$-regular multipartitions and the canonical basis elements. In the process, we also found a result on the Morita equivalence classes of cyclotomic Hecke algebras.
The reduced crystal

We get the *reduced* crystal [AS] with vertices $P(\Lambda)$ by adding edges wherever there is an edge in the underlying Kashiwara crystal, where we take all $i$-strings parallel to each other. The weights in $P(\Lambda)$ are of the form $\lambda = \Lambda - \alpha$ for some $\alpha \in Q_+$. The highest-weight representation being integrable, all $i$-strings are of finite length. To each vertex of $P(\Lambda)$ we associate

- The content $(c_0, \ldots, c_\ell)$ of $\alpha = c_0\alpha_0 + \cdots + c_\ell\alpha_\ell$
- The defect $\text{def}(\lambda) = (\Lambda \mid \alpha) - \frac{1}{2}(\alpha \mid \alpha)$
- The hub $\theta = (\theta_0, \ldots, \theta_\ell)$, where $\theta_i = \langle h_i, \lambda \rangle$

The vertices of defect zero are those equivalent to $\Lambda$ under the action of the Weyl group $W$. 
Reduced crystal, $e = 2, \Lambda = 3\Lambda_0 + 3\Lambda_1$, with hubs

The reduced crystal for $e = 2, \Lambda = 3\Lambda_0 + 3\Lambda_1$, truncated at degree 13
Chuang and Rouquier [CR] categorification:

- $V(\Lambda) \leftrightarrow \bigoplus \text{Mod}(H_n^\Lambda)$, $n = 0, 1, 2, \ldots$
- $e_i, f_i \leftrightarrow$ restriction and induction functors $E_i, F_i$
- Weight spaces $\leftrightarrow$ blocks,
- $s_i \in W \leftrightarrow$ derived equivalences,
- $s_i$ acting on end-points of $i$-string $\leftrightarrow$ Morita equivalence
- $b \in B(\Lambda) \leftrightarrow$ simple modules of $H_n^\Lambda$

The simplest but best known example is for $r = 1$, $\Lambda = \Lambda_0$, over a field of characteristic $e$, where the simple modules of the symmetric groups correspond to $e$-regular partitions.
The Fundamental Region

We want to show that for a given defect $d$ there are only a finite number of Morita equivalence classes of cyclotomic Hecke algebras, relying on categorification and using combinatorics. So far we can to this for ranks $e = 2, 3$. We start with the following theorem.

Theorem (Barshavsky, Fayers, S., 2013) For any $\eta = \Lambda - \sum_{i=0}^{\ell} b_i \alpha_i$ there is a unique $s$ such that

$$\eta - s\delta \in \text{max}(\Lambda)$$

The proof, which is given for any type of affine Lie algebra, depends on finding a fundamental region in $P(\Lambda)$ from which every element of $\text{max}(\Lambda)$ can be reached by transformations in the normal abelian subgroup $T$ in the decomposition of the Weyl group as

$$W = T \rtimes W^\circ$$
The elements of $T$ are transformations of the form

$$t_{\alpha}(\zeta) = \zeta + r\alpha - ((\zeta|\alpha) + \frac{1}{2}(\alpha|\alpha)r)\delta$$

By the theorem in the previous slide, every vertex in $\text{max}(\Lambda)$ is equivalent by the action of the Weyl subgroup $T$ to a point in the fundamental region and every defect is congruent mod $r$ to a defect in the fundamental region. Thus the elements of $\text{max}(\Lambda)$ correspond one-to-one to points of the integral lattice generated by $\alpha_1, \ldots, \alpha_\ell$.

On a string, $\text{def}(\lambda - k\alpha_i)$ is parabolic in $k$, so the defects rise to the center, and for a string of length $c$ there can be no defect less than $c$ except at the ends. Lengths of $i$-strings are determined by $\theta_i$.
The reduced crystal for $e = 3, \Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$
Maximal Strings

Proposition

For any defect $d$ in a rank 2 or 3 crystal, there is a degree $N(d)$ such that every occurrence of that defect in degree more than $N(d)$ is at the end of a string to a vertex of lower degree.

Proof.

Case $e = 3$: Let $c = r + 2d$. Let $N$ be the maximal degree occurring in a triangle $[c, -d, -d], [-d, c, -d], [-d, -d, c]$. Every hub on or outside the triangle contains a negative $\theta_i$ with $\theta_i \leq -d$. Every vertex inside the triangle has degree lower than $N + 1$. Let $\bar{d}$ be the residue of $d$ mod $r$ and set $N(d) = N + 1 + (d - \bar{d})$. Thus every $b \in B(\Lambda)$ of defect $d$ lies on a string leading to a lower degree of length $\geq d$. Since a basis element of defect $d$ cannot be internal, it is at the end of this string.
The canonical basis elements correspond to $e$-regular multipartitions. In order to generate the $e$-regular multipartitions, we must choose an ordering of the fundamental weights in $\Lambda$,

$$\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r}$$

We will follow Mathas in [M] in requiring $k_1 \leq k_2 \leq \cdots \leq k_r$, where the number of terms, $r$, in the sum is the level. We can then summarize by setting

$$\Lambda = a_0 \Lambda_0 + \cdots + a_\ell \Lambda_\ell$$

The Young diagram of a defect 0 weight $\lambda$ will be represented by $Y(\lambda)$. If the $m$-th partition of $\lambda$ is nonempty, then we associate to each node in the Young diagram a residue, where the node $(i, j)$ is given residue

$$k_m + j - i$$
Involution of multipartitions

In [Fa], Fayers describes two involutions on the multipartitions:

**Definition**

If $\lambda = (\lambda^1, \ldots, \lambda^r)$ is a multipartition of rank $e$ and level $r$, then the *conjugate* $\lambda'$ of $\lambda$ is given by $\lambda' = (\lambda^{r'}, \ldots, \lambda^{1'})$, where $\lambda^{i'}$ is the transposed partition of $\lambda^i$, corresponding to reflection of the Young diagram in the main diagonal.

**Definition**

If $\lambda = (\lambda^1, \ldots, \lambda^r)$ is a multipartition of rank $e$ and level $r$ for $\Lambda = \Lambda_{k_1} + \ldots + \Lambda_{k_r}$, then the *diamond* $\lambda^{\diamond}$ of $\lambda$ is a multipartition in the crystal for $\hat{\Lambda} = \Lambda_{-k_1} + \ldots + \Lambda_{-k_r}$, whose path is obtained from a path giving $\lambda$ by replacing each residue by minus that residue.
Fock Space

- \( \mathcal{U} = \mathcal{U}_\nu(\hat{\mathfrak{sl}}(e)) \): quantum enveloping algebra over \( \mathbb{Q}(\nu) \)
- \([n]_\nu = \nu^{n-1} + \nu^{n-3} + \cdots + \nu^{-(n-3)} + \nu^{-(n-1)}\).
- Generators: \( e_i, f_i, h_i \) for \( i \in I = \mathbb{Z}/\mathbb{Z}e \) and a central \( c \).
- If \( \Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r} \), set \( s = (k_1, \ldots, k_r) \).

The Fock space \( \mathcal{F}^s \) is a space with basis given by multipartitions consisting of \( r \) partitions. An addable \( i \)-node \( n \) is a node outside \( \lambda \) such that if added it would give a multipartition \( \lambda^n \) and would have residue \( i \). A removable \( i \)-node \( m \) inside a multipartition \( \mu \) is a node at the end of a row or column which would give a multipartition \( \mu_m \) if removed.
Fock Space

The quantum enveloping algebra $\mathcal{U}_\gamma(\hat{sl}(e))$ acts on the Fock space by determining actions for the elements of the Chevalley basis, as follows:

- For an addable node, define $N(n, i) = \#\{\text{addable } i\text{-nodes above } n\} - \#\{\text{removable } i\text{-nodes above } m\}$ and set
  \[
  f_i(\lambda) = \sum_n \nu^{N(n,i)} \lambda^n.
  \]

- For a removable node, define $M(m, i) = \#\{\text{addable } i\text{-nodes below } m\} - \#\{\text{removable } i\text{-nodes below } m\}$.
  \[
  e_i(\mu) = \sum_m \nu^{M(m,i)} \mu_m.
  \]
The divided powers are $e_i^{(k)}$ and $f_i^{(k)}$ and they are given by dividing by the quantum factorials $[k]_v!$. We define $\mathcal{F}_A^s$ to be the subalgebra of $\mathcal{F}^s$ generated by the divided powers from the highest weight vector over $A$, where coefficients lie in the algebra $A$ of Laurent polynomials in $v$ with integral coefficients. In addition, there is an involution of the quantum enveloping algebra called the bar-involution which fixes $e_i$, $f_i$ and $h_i$, but interchanges $v$ and $v^{-1}$. 
Canonical basis

For each e-regular multipartition $\mu$, there is an element $G(\mu)$ of the Fock space $\mathcal{F}_A^s$ that is invariant under the bar involution, and these are called the canonical basis elements. The action of the Chevalley basis elements $e_i$ and $f_i$ on these canonical basis elements is induced from their action on the basis elements of the Fock space. We write

$$G(\mu) = \sum_{\lambda \in P^r} d_{\lambda \mu}(v) \lambda$$
Definition

For a sequence $S$ chosen from a two element ordered set $\{0, 1\}$, the number of inversions, $\text{Inv}(S)$, is the sum of the number of elements 0 appearing before each element 1.

Definition

The shape of a canonical basis element is the number of multipartitions, counting repetitions, for each power of $v$.

Definition

A vertex $v$ in the reduced crystal is $i$-external if $\tilde{e}_i(b)$ is zero for every element of $B(\Lambda)$ with the weight corresponding to $v$. 
Lemma

Let \( \mu \) be an \( e \)-regular multipartition for an \( i \)-external vertex of the reduced crystal with \( i \)-string of length \( c \). Let \( S(c, k) \) be the set of sequences of length \( c \) with \( k \) copies of 1 and \( c - k \) copies of 0. For any \( S \in S(c, k) \), let \( \lambda^S \) be the multipartition obtained by adding each node corresponding to a 1. If every multipartitions occurring in \( G(\lambda) \) has \( c \) of addable \( i \)-nodes and no removable nodes, then

\[
\tilde{f}_{i}^{(k)}(G(\mu)) = \sum_{S \in S(c, k)} \sum_{\lambda \in P^r} d_{\lambda \mu} v^{\text{Inv}(S)} \lambda^S
\]

For \( k = c \) and \( S \) the unique sequence with all copies of 1,

\[
G(\mu^S) = \sum_{\lambda \in P^r} d_{\lambda \mu} \lambda^S
\]
In Theorem 2.1 of [Fa], Fayers proves that if $w(\mu)$ is the defect of an $e$-regular multipartition $\mu$, then

$$\hat{d}_{\lambda', \mu} = v^{w(\mu)} d_{\lambda, \mu}(v^{-1}).$$

This theorem involves constructing two distinct crystals and comparing them. Instead, let $\Lambda = a\Lambda_0 + a\Lambda_1 + \cdots + a\Lambda_\ell$ be symmetric. By using Fayer's $\diamond$ involution, followed by a reshuffling of the tuple $s$ used in the definition of the Fock space, we find that for any path, the path we get by exchanging $k$ with $\ell - k$ has canonical basis elements which by symmetry have the same shape and by the $\diamond$ involution are thus symmetric.
Let us take a completely symmetric crystal, 
\[ \Lambda = a\Lambda_0 + a\Lambda_1 + \cdots + a\Lambda_\ell. \]

- The canonical basis elements of a symmetric crystal have a symmetric shape.
- For \( e = 2 \), we can give formulae for the canonical basis elements of all weights with defect \( a(k - a) \).
- For \( e = 3 \), we can give formula for canonical basis elements with for weights \( \lambda - k\alpha_i \) with \( 0 \leq k \leq a \). Every block of the cyclotomic Hecke algebra is Morita equivalent to one of a finite set of blocks of degree less than some bound \( N(d) \) and the canonical basis elements are easily obtained from those of this finite set.
The Littelmann paths for a given $G$ and $\Lambda$ can be generated in Sagemath using the function `CrystalOfLSPaths()` written by Mark Shimozono and Anne Schilling. In addition, Travis Scrimshaw implemented an algorithm of Matt Fayers to calculate the canonical basis, named `FockSpace()`.

Our own modification computes the following for a basis element $b \in B(\Lambda)$:

- The multipartition
- The Littelmann path and, optionally, the corner-points
- The canonical basis element
- The set of paths in the reduced crystal leading to $b$
Bibliography

S. Ariki, V. Kreiman, & S. Tsuchioka, *On the tensor product of two basic representations of \( U_v(\hat{sl}_e) \)*, Advances in Mathematics 218 (2008), 28-86.


Thank you.
Thank you.

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